Reliable computation in geotechnical stability analysis

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Introduction

Geotechnical stability includes:

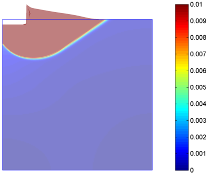
- stability of slopes, foundations, tunnels, excavations, etc.

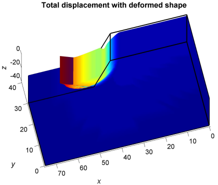
Aims of geotechnical stability analysis:

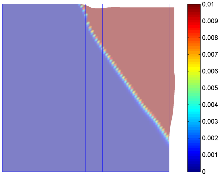
- safety factor for a given set of applied loads and material parameters
    - strength or limit load parameters
- failure mechanisms caused by limit (ultimate) loads

Basic methods:

- slip-line method, limit equilibrium, strength reduction, incremental methods, limit analysis







History of limit analysis:

- developped by D.C. Drucker in 50' - lower and upper bound theorems
- based on perfect plasticity & associative plastic flow rule (classical theory)
- analytical methods: [W.-F. Chen: Limit analysis in soil mechanics, 1975]
- survey article: [S. Sloan: Geotechnical stability analysis. Géotechnique, 2013]

Mathematical theory of classical limit analysis:

- [R. Temam: Mathematical Problems in Plasticity. Gauthier-Villars, 1985],
- [E. Christiansen: Limit analysis of colapse states, 1996],
- [S. Repin, G. Seregin: Existence of a weak solution of the minimax problem arising in Coulomb-Mohr plasticity, 1995]

Nonclassical limit analysis:

- nonassociative plasticity with hardening/softening, porous materials
- variational approach based on theory of bipotentials
- [Zouain, Filho, Borges, da Costa 2007], [Hamlaoui, Oueslati, de Saxcé 2017]

Outline

1. Limit analysis problem for the Drucker-Prager yield criterion.
2. Inf-sup condition related to the limit analysis and computable majorant of the limit load
3. Computational strategy and mesh adaptivity.
4. Numerical examples.

1. Limit analysis problem for the Drucker-Prager yield criterion

Assumptions and notation:

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ – bounded domain with Lipschitz boundary $\partial\Omega$
- $\partial\Omega = \bar{\Gamma}_D \cap \bar{\Gamma}_N$:
  - Γ_D – homogeneous Dirichlet boundary conditions
  - Γ_N – Neumann boundary conditions
- homogeneous material
- basic functional spaces (L^2 and $W^{1,2}$)

Variational setting of the problem - notation

Space of displacement (velocity) fields:

$$\mathbb{V} = \left\{ \boldsymbol{v} \in W^{1,2}(\Omega, \mathbb{R}^d) \;\middle|\; \boldsymbol{v} = \mathbf{0} \text{ a.e. on } \Gamma_D \right\}$$

Load functional:

$$L(\boldsymbol{v}) = \int_\Omega \boldsymbol{F} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{s} \quad \forall \boldsymbol{v} \in \mathbb{V}, \quad \boldsymbol{F} \in L^2(\Omega, \mathbb{R}^d), \;\; \boldsymbol{f} \in L^2(\Gamma_N; \mathbb{R}^d)$$

Statically admissible stresses for $\lambda \geq 0$:

$$\begin{aligned} Q_{\lambda L} &= \left\{ \boldsymbol{\tau} \in L^2(\Omega, \mathbb{R}^{d\times d}_{sym}) \mid \operatorname{Div} \boldsymbol{\tau} + \lambda \boldsymbol{F} = 0 \text{ in } \Omega, \;\; \boldsymbol{\tau}\boldsymbol{\nu} = \lambda \boldsymbol{f} \text{ on } \Gamma_N \right\} \\ &= \left\{ \boldsymbol{\tau} \in L^2(\Omega, \mathbb{R}^{d\times d}_{sym}) \mid \int_\Omega \boldsymbol{\tau} : \varepsilon(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} = \lambda L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbb{V} \right\}. \end{aligned}$$

Plastically admissible stresses:

$$P = \left\{ \boldsymbol{\tau} \in L^2(\Omega, \mathbb{R}^{d\times d}_{sym}) \mid \boldsymbol{\tau}(\boldsymbol{x}) \in B \text{ for a.a. } \boldsymbol{x} \in \Omega \right\}, \quad B \subset \mathbb{R}^{d\times d}_{sym} - \text{convex}$$

Problem of limit analysis

Static approach (lower bound theorem of limit analysis):

$$\lambda^* = \sup\{\lambda \geq 0 \mid \; Q_{\lambda L} \cap P \neq \emptyset\}$$

Kinematic approach (upper bound theorem of limit analysis):

$$\zeta^* = \inf_{\substack{\boldsymbol{v} \in \mathbb{V} \\ L(\boldsymbol{v})=1}} \int_\Omega j_\infty(\varepsilon(\boldsymbol{v}))\,\mathrm{d}\boldsymbol{x}, \qquad j_\infty(\boldsymbol{e}) := \sup_{\boldsymbol{\tau} \in B} \boldsymbol{\tau} : \boldsymbol{e}, \quad \boldsymbol{e} \in \mathbb{R}^{d\times d}_{sym}$$

Duality within limit analysis:

$$\lambda^* = \sup_{\substack{\boldsymbol{\tau} \in L^2(\Omega,\mathbb{R}^{d\times d}_{sym}) \\ \boldsymbol{\tau} \in B \text{ in } \Omega}} \; \inf_{\substack{\boldsymbol{v} \in \mathbb{V} \\ L(\boldsymbol{v})=1}} \int_\Omega \boldsymbol{\tau} : \varepsilon(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} \leq \inf_{\substack{\boldsymbol{v} \in \mathbb{V} \\ L(\boldsymbol{v})=1}} \int_\Omega j_\infty(\varepsilon(\boldsymbol{v}))\,\mathrm{d}\boldsymbol{x} = \zeta^*,$$

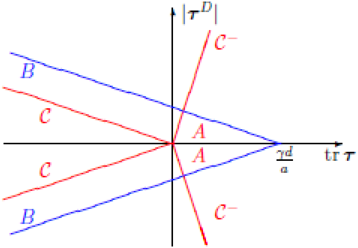
Comments:

- $\lambda^* \leq \zeta^*$ but the equality $\lambda^* = \zeta^*$ can be investigated
- a minimum defining ζ^* need not belong to $\mathbb{V}$
  $\Longrightarrow$ relaxation of the problem, *BD*-spaces

The Drucker-Prager yield criterion

Set B including the D-P yield criterion:

- $B = \{\boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{sym} \mid |\boldsymbol{\tau}^D| + \frac{a}{d}\mathrm{tr}\,\boldsymbol{\tau} \leq \gamma\}$,
  $a, \gamma > 0$ – material parameters
- $\mathcal{C} = \{\boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{sym} \mid |\boldsymbol{\tau}^D| + \frac{a}{d}\mathrm{tr}\,\boldsymbol{\tau} \leq 0\}$,
- $\mathcal{C}^- = \{\boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{sym} \mid \boldsymbol{\eta} : \boldsymbol{\tau} \leq 0 \ \ \forall \boldsymbol{\tau} \in \mathcal{C}\}$
- $\mathcal{C}^- = \{\boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{sym} \mid \mathrm{tr}\,\boldsymbol{\eta} \geq a|\boldsymbol{\eta}^D|\}$



Consequence: $j_\infty(\boldsymbol{e}) = \sup_{\boldsymbol{\tau} \in B} \boldsymbol{\tau} : \boldsymbol{e} = \frac{\gamma}{a}\mathrm{tr}\,\boldsymbol{e}$ if $\boldsymbol{e} \in \mathcal{C}^-$, otherwise $j_\infty(\boldsymbol{e}) = +\infty$

Kinematic limit analysis problem $(\mathcal{P})^\infty$:

$$(\mathcal{P})^\infty \qquad \zeta^* = \inf_{\substack{\boldsymbol{v} \in \mathcal{K} \\ L(\boldsymbol{v})=1}} \int_\Omega \frac{\gamma}{a} \mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}, \qquad \mathrm{div}\,\boldsymbol{v} = \mathrm{tr}\,\varepsilon(\boldsymbol{v}),$$

$$\mathcal{K} = \{\boldsymbol{w} \in \mathbb{V} \mid \varepsilon(\boldsymbol{w}) \in \mathcal{C}^- \text{ in } \Omega\} = \{\boldsymbol{w} \in \mathbb{V} \mid \ \mathrm{div}\,\boldsymbol{w} \geq a|\varepsilon^D(\boldsymbol{w})| \text{ in } \Omega\}$$

2. Inf–sup conditions related to limit analysis and computable majorant of the limit load

- **Set of constrains in kinematic limit analysis:**

$$\mathcal{K} = \{\boldsymbol{w} \in \mathbb{V} \mid \varepsilon(\boldsymbol{w}) \in \mathcal{C}^- \text{ a.e. in } \Omega\}$$

- **Related sets of Lagrange multipliers:**

$$L^2(\Omega;\mathcal{C}) = \{\boldsymbol{\tau} \in L^2(\Omega;\mathbb{R}^{d\times d}_{sym}) \mid \boldsymbol{\tau} \in \mathcal{C} \text{ a.e. in } \Omega\}$$

$$\boldsymbol{w} \in \mathcal{K} \quad\iff\quad \int_\Omega \boldsymbol{\tau} : \varepsilon(\boldsymbol{w})\,\mathrm{d}\boldsymbol{x} \leq 0 \quad \forall \boldsymbol{\tau} \in L^2(\Omega;\mathcal{C})$$

- L^2**-norm of scalar, vector and tensor functions in** Ω**:** $\|.\|_\Omega$

Distance estimate of functions to the set $\mathcal{K}$

Assumption:

$$c_* := \inf_{\substack{\boldsymbol{\tau}\in L^2(\Omega;\mathcal{C})\\ \boldsymbol{\tau}\neq 0}} \sup_{\substack{\boldsymbol{v}\in\mathbb{V}\\ \boldsymbol{v}\neq 0}} \frac{\int_\Omega \boldsymbol{\tau} : \nabla \boldsymbol{v}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\tau}\|_\Omega \|\nabla \boldsymbol{v}\|_\Omega} > 0.$$

The distance estimate to $\mathcal{K}$:

$$\min_{\boldsymbol{w}\in\mathcal{K}} \|\nabla(\boldsymbol{v}-\boldsymbol{w})\|_\Omega \leq \frac{1}{c_*}\|\Pi_{\mathcal{C}}\,\varepsilon(\boldsymbol{v})\|_\Omega \quad \forall \boldsymbol{v}\in\mathbb{V}.$$

$\Pi_{\mathcal{C}}$ – projection of $\mathbb{R}^{d\times d}_{sym}$ onto $\mathcal{C}$ w.r.t. the biscalar product

Idea of the proof:

$$\min_{\boldsymbol{w}\in\mathcal{K}} \|\nabla(\boldsymbol{v}-\boldsymbol{w})\|^2_\Omega = \inf_{\boldsymbol{w}\in\mathbb{V}} \sup_{\boldsymbol{\tau}\in L^2(\Omega;\mathcal{C})} \left\{ \|\nabla(\boldsymbol{v}-\boldsymbol{w})\|^2_\Omega + 2\int_\Omega \boldsymbol{\tau} : \nabla \boldsymbol{w}\,\mathrm{d}\boldsymbol{x} \right\}.$$

$$c_* > 0 \quad \Longrightarrow \quad \inf\sup = \sup\inf \leq \frac{1}{c_*^2}\|\Pi_{\mathcal{C}}\,\varepsilon(\boldsymbol{v})\|^2_\Omega$$

Consequences of the distance estimate

1. Equivalence between the static and kinematic approaches:

$$\lambda^* = \zeta^* \quad \text{[Repin, Seregin 1995]}$$

2. Guaranteed upper bound of ζ^*:

$$\zeta^* \leq \frac{J^A_\infty(\boldsymbol{v})}{L(\boldsymbol{v})} \quad \forall \boldsymbol{v} \in \mathcal{K},\ L(\boldsymbol{v}) > 0, \qquad J^A_\infty(\boldsymbol{v}) := \int_\Omega \frac{\gamma}{a} \operatorname{div} \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}$$

$$\zeta^* \leq \frac{J^A_\infty(\boldsymbol{v}) + c_*^{-1} \rho_A |\Omega|^{1/2} \|\Pi_{\mathcal{C}}\, \varepsilon(\boldsymbol{v})\|_\Omega}{L(\boldsymbol{v}) - c_*^{-1} \|L\|_* \|\Pi_{\mathcal{C}}\, \varepsilon(\boldsymbol{v})\|_\Omega} \quad \forall \boldsymbol{v} \in \mathbb{V},\ L(\boldsymbol{v}) > c_*^{-1} \|L\|_* \|\Pi_{\mathcal{C}}\, \varepsilon(\boldsymbol{v})\|_\Omega$$

- there are available computable majorants of the constant $\rho_A > 0$, $\|L\|_* > 0$
- **computable** majorants of $1/c_*$ are needed
- if $\boldsymbol{v} \in \mathcal{K}$ then both estimates coincide since $\|\Pi_{\mathcal{C}}\, \varepsilon(\boldsymbol{v})\|_\Omega = 0\ \forall \boldsymbol{v} \in \mathcal{K}$

On validity of the inf–sup condition

1. One cannot investigate the inf–sup condition on the whole space!

$$\inf_{\substack{\boldsymbol{\tau}\in L^2(\Omega;\mathcal{C})\\ \boldsymbol{\tau}\neq 0}} \sup_{\substack{\boldsymbol{v}\in\mathbb{V}\\ \boldsymbol{v}\neq 0}} \frac{\int_\Omega \boldsymbol{\tau}:\nabla\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\tau}\|_\Omega\|\nabla\boldsymbol{v}\|_\Omega} \geq \inf_{\substack{\boldsymbol{\tau}\in L^2(\Omega;\mathbb{R}^{d\times d}_{sym})\\ \boldsymbol{\tau}\neq 0}} \sup_{\substack{\boldsymbol{v}\in\mathbb{V}\\ \boldsymbol{v}\neq 0}} \frac{\int_\Omega \boldsymbol{\tau}:\nabla\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\tau}\|_\Omega\|\nabla\boldsymbol{v}\|_\Omega} = 0$$

2. Dirichlet b.c. on the whole boundary: $\mathbb{V} = W_0^{1,2}(\Omega;\mathbb{R}^d)$

$\Rightarrow$ the inf-sup condition does not hold, i.e. $c_* = 0$

$\Rightarrow$ $\mathcal{K} = \{0\}$, $\lambda^* = \zeta^* = +\infty$ [Repin, Seregin 1995]

3. Dirichlet b.c. on a part of $\partial\Omega$, i.e. $\mathbb{V} \neq W_0^{1,2}(\Omega;\mathbb{R}^d)$: Let

$$a < \frac{1}{\sqrt{C_\Omega^2 - d^{-1}}}, \quad \text{where } \frac{1}{C_\Omega} := \inf_{\substack{q\in L^2(\Omega)\\ q\neq 0}} \sup_{\substack{\boldsymbol{v}\in\mathbb{V}\\ \boldsymbol{v}\neq 0}} \frac{\int_\Omega q\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}}{\|q\|_\Omega\|\nabla\boldsymbol{v}\|_\Omega} > 0.$$

Then

$$c_* = \inf_{\substack{\boldsymbol{\tau}\in L^2(\Omega;\mathcal{C})\\ \boldsymbol{\tau}\neq 0}} \sup_{\substack{\boldsymbol{v}\in\mathbb{V}\\ \boldsymbol{v}\neq 0}} \frac{\int_\Omega \boldsymbol{\tau}:\nabla\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{\tau}\|_\Omega\|\nabla\boldsymbol{v}\|_\Omega} \geq \frac{1 - a\sqrt{C_\Omega^2 - d^{-1}}}{C_\Omega\sqrt{a^2 + d}} > 0.$$

On computable majorants of the constant C_Ω

Inf-sup conditions for incompressible flow media:

$$\frac{1}{C_\Omega} = \inf_{\substack{q \in L^2(\Omega) \\ q \neq 0}} \sup_{\substack{\boldsymbol{v} \in \mathbb{V} \\ \boldsymbol{v} \neq 0}} \frac{\int_\Omega q \operatorname{div} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}}{\|q\|_\Omega \|\nabla \boldsymbol{v}\|_\Omega} > 0,$$

$$\frac{1}{C_\Omega^0} := \inf_{\substack{q \in L^2(\Omega) \\ \{q\}_\Omega = 0}} \sup_{\boldsymbol{v} \in W_0^{1,2}(\Omega;\mathbb{R}^d)} \frac{\int_\Omega q \operatorname{div} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}}{\|q\|_\Omega \|\nabla \boldsymbol{v}\|_\Omega} > 0, \quad \{q\}_\Omega := \frac{1}{|\Omega|} \int_\Omega q \, \mathrm{d}\boldsymbol{x} = 0$$

Analytical upper bounds of C_Ω^0: [Dauge, Costabel 2015], [Payne 2007]

- 2D: star-shaped domains Ω w.r.t. a concentric ball of radius $\rho > 0$
- 3D: only for some special cases

Relation between C_Ω and C_Ω^0:

$$C_\Omega \leq \sqrt{(C_\Omega^0)^2 + |\Omega|^{-1} \|\nabla \tilde{v}\|_\Omega^2} \quad \forall \tilde{v} \in \mathbb{V}, \ \operatorname{div} \tilde{v} = 1 \text{ in } \Omega$$

Semianalytical bounds of C_Ω: [3x Repin 2015-8], [Repin, S., Haslinger 2018]

3. Computational strategy and mesh adaptivity

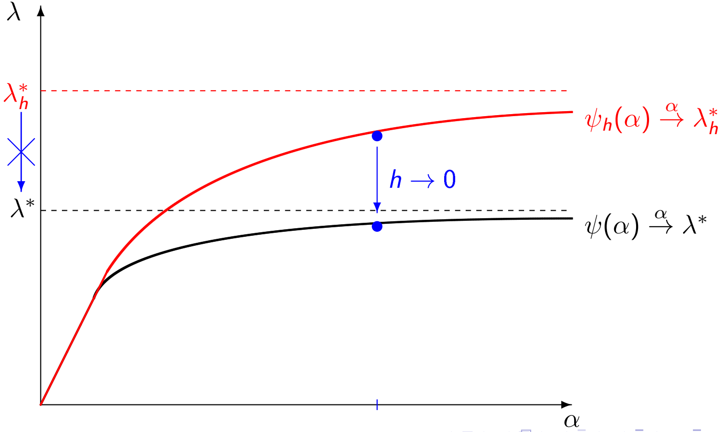
- **Penalization:**

$$\lambda_\alpha = \inf_{\substack{\boldsymbol{v}\in\mathbb{V} \\ L(\boldsymbol{v})=1}} \int_\Omega j_\alpha(\varepsilon(\boldsymbol{v}))\,dx, \quad j_\alpha(\boldsymbol{e}) = \sup_{\boldsymbol{\tau}\in B} \{\boldsymbol{\tau} : \boldsymbol{e} - \frac{1}{2\alpha}\mathbb{C}^{-1}\boldsymbol{\tau} : \boldsymbol{\tau}\},$$

  - $\alpha > 0$, j_α – real-valued and smooth, $j_\alpha \to j_\infty$ as $\alpha \to +\infty$
  - $\mathbb{C}$ – elastic fourth order tensor
  - closely related to an elastic-perfectly plastic problem
- **Discretization: standard finite element method**
- **Solver: continuation through α & semismooth Newton method**
- **Implementation: vectorized Matlab codes in 2D and 3D**
- M. Čermák, S. Sysala, J. Valdman: Fast MATLAB assembly of elastoplastic FEM matrices in 2D and 3D. ArXiv 1805.04155. Codes available in `https://github.com/matlabfem/matlab_fem_elastoplasticity`

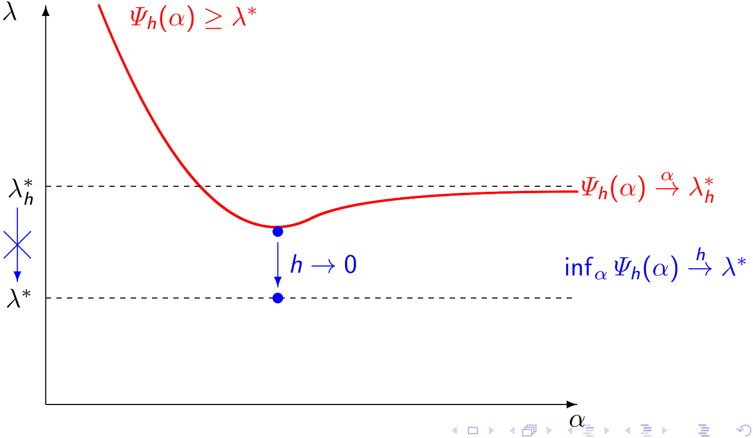
Output 1: load path function ψ_h

$$\alpha \mapsto \boldsymbol{u}_{h,\alpha} \mapsto \psi_h(\alpha) := \int_\Omega \Pi_B(\varepsilon(\alpha \boldsymbol{u}_{h,\alpha})) : \varepsilon(\boldsymbol{u}_{h,\alpha})\,\mathrm{d}\boldsymbol{x}$$



Output2: upper bound function Ψ_h

$$\alpha \mapsto \boldsymbol{u}_{h,\alpha} \mapsto \Psi_h(\alpha) := \frac{J_\infty^A(\boldsymbol{u}_{h,\alpha}) + c_*^{-1}\rho_A|\Omega|^{1/2}\|\Pi_{\mathcal{C}}\,\varepsilon(\boldsymbol{u}_{h,\alpha})\|_\Omega}{L(\boldsymbol{u}_{h,\alpha}) - c_*^{-1}\|L\|_*\|\Pi_{\mathcal{C}}\,\varepsilon(\boldsymbol{u}_{h,\alpha})\|_\Omega}$$



Mesh adaptivity

Reasons:

- failure is localized - estimation of the failure surface
- presence of the rigid kinematic fields far from the failure

Restriction on mesh refinement:

- $P1$ and $P2$ elements in 2D
- all triangles must isoscaled and right-angled before and after refinements
- the longest-edge bisection on selected triangles

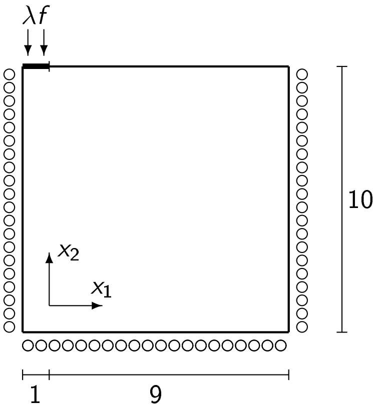
Adaptive strategy

1. fix penalization parameter α sufficiently large
2. solve $(\mathcal{P})_h^\alpha$: minimize $\int_\Omega j_\alpha(\varepsilon(\boldsymbol{v}_h))\,\mathrm{d}\boldsymbol{x}$ s.t. $L(\boldsymbol{v}_h) = 1 \quad \mapsto \boldsymbol{u}_{h,\alpha}$
3. sort the array $\{\int_e \operatorname{div} \boldsymbol{u}_{h,\alpha}\,\mathrm{d}\boldsymbol{x}, \quad e \in \mathcal{T}_h\}$ (see problem $(\mathcal{P})^\infty$)
4. select a smaller number of elements $e \in \mathcal{T}_h$ from the sorted array
5. refine mesh according to the prescribed sets of elements
6. interpolate $\boldsymbol{u}_{h,\alpha}$ to the refined mesh to initiate a Newton-like solver

Comments:

- problem $(\mathcal{P})_h^\alpha$ is strongly nonlinear for larger α
  - $\Rightarrow$ continuation Newton method for the initial mesh
  - $\Rightarrow$ damped Newton method for refined meshes
  - $\Rightarrow$ more refinements with a smaller number of refine elements
- Adaptive Newton method [Axelsson, Sysala 2018 - submitted]

Example 1 – Strip footing & bearing capacity

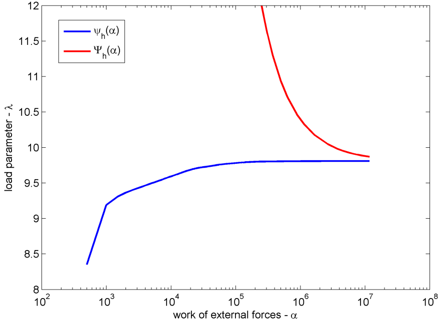


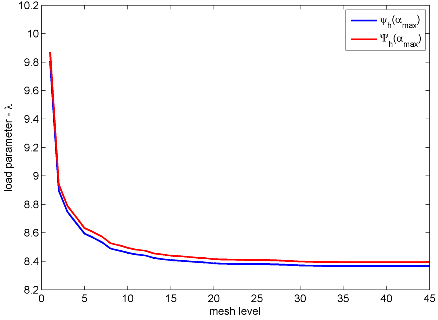
- plane strain problem $(d = 3)$
- friction angle $= 10^o$, cohesion $= 450$
- hence $a = 0.24$, $\gamma = 624$
- $f = -450$, $\|L\|_* \leq 450\sqrt{10}$
- $C^0_\Omega \leq 2.6$, $C_\Omega \leq 2.8$, $c_*^{-1} \leq 14.8$
- P2 elements, 7-point Gauss quadrature

Load path & guaranteed upper bound

- results for initial (coarsest) mesh
- $\psi_h(\alpha)$ and $\Psi_h(\alpha)$, $\alpha \in (0, 10^7)$
- α - convergence

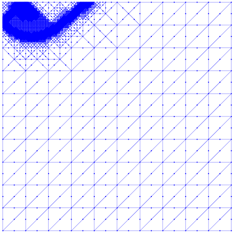
- 45 levels of mesh refinement
- $\psi_h(\hat{\alpha})$ and $\Psi_h(\hat{\alpha})$, $\hat{\alpha} \doteq 10^7$
- h-convergence

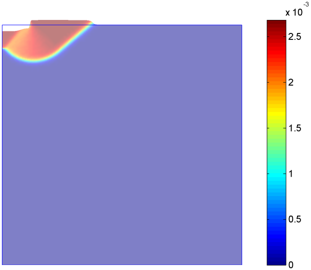


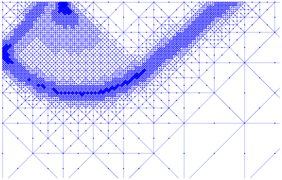


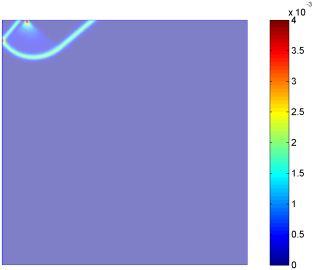
$$\lambda^* = \zeta^* \leq \Psi_h(\hat{\alpha}) \doteq 8.39, \quad \lambda^* = \zeta^* \geq \lim_{h \to 0_+} \psi_h(\hat{\alpha}) \doteq 8.36$$

Visualization of failure - the finest mesh

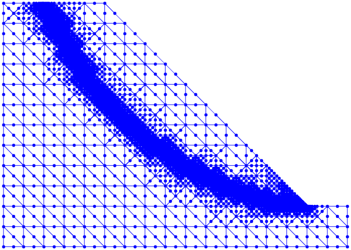


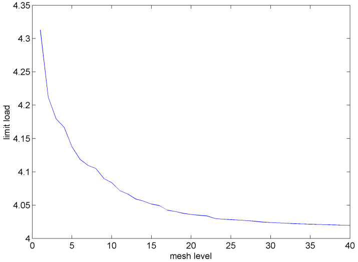


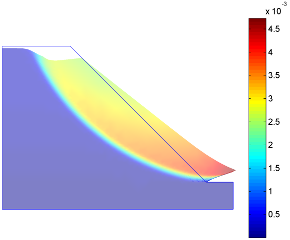


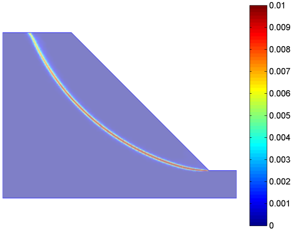


Example 2 – Slope stability









Concluding remark 1

Restrictive assumption on the guaranteed upper bound:

$$a < \frac{1}{\sqrt{C_\Omega^2 - d^{-1}}}$$

- material parameter a is dependent on the domain Ω!
- one must set too small friction angle
- the upper bound is sharp only for very large α

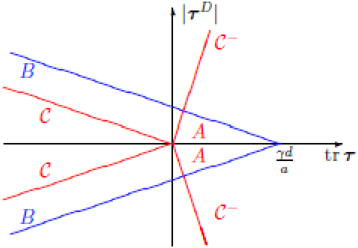
Deeper analysis of inf–sup conditions on convex cones:

- absence of literature on this problematic
- possible applications out of plasticity
- equivalent statements to the inf–sup condition?

Concluding remark 2

Extension of the results for other yield criteria – assumptions on B:

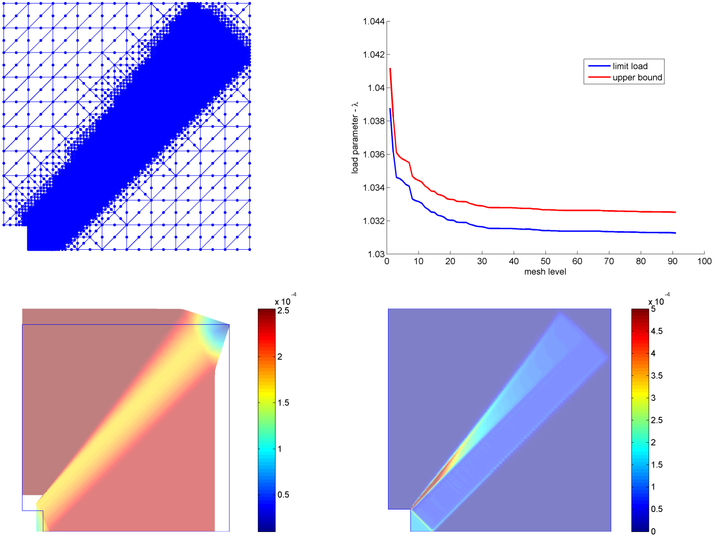
- $B \subset \mathbb{R}^{d \times d}_{sym}$ is closed, convex
- $\mathbf{0} \in \operatorname{int} B$
- $B = \mathcal{C} + A$,
  - C – convex cone with vertex at $\mathbf{0}$,
  - $A \subset \mathcal{C}^-$ – bounded



Examples of yield criteria satisfying the assumptions:

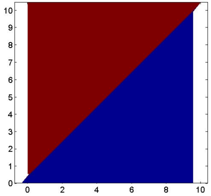
- von Mises, Tresca, Drucker-Prager, Mohr-Coulomb

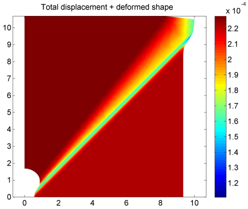
Example 3 – von Mises yield criterion



Our recent articles containing limit analysis:

1. S. Sysala, J. Haslinger, **I. Hlaváček**, M. Cermak: *Discretization and numerical realization of contact problems for elastic-perfectly plastic bodies. PART I - discretization, limit analysis*. ZAMM 95 (2015) 333 – 353.
2. M. Cermak, J. Haslinger, T. Kozubek, S. Sysala: *Discretization and numerical realization of contact problems for elastic-perfectly plastic bodies. PART II - numerical realization, limit analysis*. ZAMM 95 (2015) 1348–371.
3. J. Haslinger, S. Repin, S. Sysala: *A reliable incremental method of computing the limit load in deformation plasticity based on compliance: Continuous and discrete setting*. Journal of Computational and Applied Mathematics 303 (2016) 156–170.
4. J. Haslinger, S. Repin, S. Sysala: *Guaranteed and computable bounds of the limit load for variational problems with linear growth energy functionals*. Applications of Mathematics 61 (2016) 527–564.
5. S. Sysala, M. Cermak, T. Koudelka, J. Kruis, J. Zeman, R. Blaheta: Subdifferential-based implicit return-mapping operators in computational plasticity. ZAMM 96 (2016) 1318–1338.
6. S. Sysala, M. Čermák, T. Ligurský: *Subdifferential-based implicit return-mapping operators in Mohr-Coulomb plasticity*. ZAMM 97 (2017) 1502–1523.
7. S. Sysala, J. Haslinger: *Truncation and Indirect Incremental Methods in Hencky's Perfect Plasticity*. In: Mathematical Modelling in Solid Mechanics (265–284), 2017. Springer.
8. S. Repin, S. Sysala, J. Haslinger: *Computable majorants of the limit load in Hencky's plasticity problems*. Computer & Mathematics with Applications 75 (2018) 199–217.
9. J. Haslinger, S. Repin , S. Sysala: *Stress-displacement inf–sup conditions on convex cones with applications to perfect plasticity*. In preparation.





Thank you for your attention!